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Deduction of L. Hörmander's extension of Ásgeirsson's mean value theorem

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Abstract

L. Hörmander's extension of Ásgeirsson's mean value theorem states that if u is a solution of the inhomogeneous ultrahyperbolic equation $(\Delta_x - \Delta_y)u = f$, $x, y \in \mathbf{R}^\nu$, $f \in \mathcal{E}'(\mathbf{R}^{2\nu})$, then

$$[\delta(t-|x|)\otimes\delta(y) - \delta(x)\otimes\delta(t-|y|)] * u = \mu_t * f,$$

where t is positive and μ_t is given explicitly. Whereas L. Hörmander proves this by defining μ_t without much motivation and then verifying the equation

$$(\Delta_x - \Delta_y)\mu_t = \delta(t-|x|)\otimes\delta(y) - \delta(x)\otimes\delta(t-|y|),$$

the aim of this article is a *constructive* deduction of a slightly different representation of the distribution μ_t, viz. μ_t is the analytic continuation of

$$\mu_t^z := \frac{Y(t-|x|-|y|)}{2\Gamma((3-z)/2)}\left(\frac{\pi A}{4t^2}\right)^{(1-z)/2} \in L^1_{\mathrm{loc}}(\mathbf{R}^{2\nu}) \subset \mathcal{D}'(\mathbf{R}^{2\nu}), \quad \operatorname{Re} z < 3,$$

into the value $z = \nu$, where $A = t^4 - 2t^2(|x|^2+|y|^2) + (|x|^2-|y|^2)^2$.

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1. Introduction

The classical mean value property for a harmonic function u, i.e.,

$$u(0) = \frac{1}{R^{n-1}\omega_n} \int\limits_{|x|=R} u(x)\,\mathrm{d}o(x), \tag{1}$$

$\omega_n = 2\pi^{n/2}/\Gamma(n/2)$, $\mathrm{d}o =$ surface measure, can be formulated as convolution equation: $H_R * u = 0$ for u with $\Delta u = 0$ if $H_R := \delta(R - |x|) - R^{n-1}\omega_n\delta \in \mathcal{E}'(\mathbf{R}^n)$ [13, p. 217].

L. Schwartz has shown that, for a general $H \in \mathcal{E}'(\mathbf{R}^n)$, the property $H * u = 0$ for all u with $\Delta u = 0$ is equivalent to the (unique) existence of $\mu \in \mathcal{E}'(\mathbb{R}^n)$ such that

$$\Delta\mu = H$$

(cf. [13, p. 217]). The knowledge of μ then yields the extension of the mean value property (1) for solutions of the homogeneous equation ($\Delta u = 0$) to the following property (2) for solutions u of the inhomogeneous equation:

$$H * u = \mu * f \quad \text{for } u \text{ with } \Delta u = f \in \mathcal{E}'. \tag{2}$$

In the case of $H = H_R$, μ is given explicitly in [2, IV, § 3,1, p. 277(6′)], (for $n \neq 2$) by the expression $\mu = \frac{R^{n-1}}{n-2}(|x|^{2-n} - R^{2-n})\chi_+^0(R - |x|)$ where χ_+^λ denotes the analytic distribution-valued function

$$\chi_+^\lambda : \mathbf{C} \to \mathcal{D}'(\mathbf{R}_t^1) : \lambda \mapsto \frac{t_+^\lambda}{\Gamma(\lambda+1)}$$

([6, pp. 56, 57]; [13, p. 43, (II,2;31)]; [9, (2.2.5) Ejemplo, pp. 85–88]; [3, p. 266 (17.9.3.3)]; [7, p. 73, (3.2.17)]). The analogue of (1), (2) is valid for $H \in \mathcal{E}'$ and a partial differential operator $P(\partial)$ with constant coefficients, i.e., $H * u = 0$ for all $u \in \mathcal{D}'$ with $P(\partial)u = 0$ if and only if $\exists!\mu \in \mathcal{E}'$: $P(\partial)\mu = H$, and hence $H * u = \mu * f$ for all $u \in \mathcal{D}'$ with $P(\partial)u = f \in \mathcal{E}'$ (cf. [11, Proposition 2, p. 291]).

In 1937, L. Ásgeirsson found an analogue of (1) with respect to the ultrahyperbolic operator $\Delta_x - \Delta_y$ in $\mathbf{R}^{2\nu}_{x,y}$, namely

$$\int\limits_{|x|=R} u(x,0)\,\mathrm{d}o(x) = \int\limits_{|y|=R} u(0,y)\,\mathrm{d}o(y) \quad \text{if}$$

$$(\Delta_x - \Delta_y)u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_\nu^2} - \frac{\partial^2 u}{\partial y_1^2} - \cdots - \frac{\partial^2 u}{\partial y_\nu^2} = 0$$

(cf. [1]; [2, p. 746 (5)]).

Hence there must exist $\mu_R \in \mathcal{E}'(\mathbf{R}^{2\nu})$ fulfilling

$$(\Delta_x - \Delta_y)\mu_R = \widetilde{H}_R := \delta\big(R - |x|\big) \otimes \delta(y) - \delta(x) \otimes \delta\big(R - |y|\big).$$

For $\nu = 3$, μ_R was found explicitly by H. Lewy [10], whereas, for general ν, a formula for μ_R was given only recently by Hörmander (cf. [8, Theorem 2.1, p. 380]).

The aim of this note is to deduce *constructively* the distribution μ_R, which in [8] was just verified to solve $(\Delta_x - \Delta_y)\mu_R = \widetilde{H}_R$. Moreover, we define μ_R here differently as the value at $z = \nu$ of a meromorphic distribution-valued function $z \mapsto \mu_R^z$ that – similarly to the M. Riesz kernels – fulfills a recurrence relation with respect to the ultrahyperbolic operator.

Notations

In general, we adopt the notations of L. Schwartz's book "Théorie des distributions". In particular, $Y = \chi_+^0$ denotes the Heaviside function and $*$ means convolution. We consider $x, y \in \mathbf{R}^\nu$ and thus $(t, x, y) \in \mathbf{R}^{1+2\nu}$ and write $|x| = (x_1^2 + \cdots + x_\nu^2)^{1/2}$, $\Delta_x = \sum_{j=1}^\nu \frac{\partial^2}{\partial x_j^2}$, $\Delta_y = \sum_{j=1}^\nu \frac{\partial^2}{\partial y_j^2}$, $\Box = \partial_t^2 - \Delta_x$. We use the Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbf{R}^{1+2\nu}) \to \mathcal{S}'(\mathbf{R}^{1+2\nu})$ in the form

$$\mathcal{F}\phi = \int_{\mathbf{R}^{1+2\nu}} \phi(\tau, \xi, \eta)\, \mathrm{e}^{-\mathrm{i}(t\tau + x\xi + y\eta)}\, \mathrm{d}\tau\, \mathrm{d}\xi\, \mathrm{d}\eta, \quad \phi \in \mathcal{S}\big(\mathbf{R}^{1+2\nu}\big).$$

2. Representation of μ by a parameter integral

As described in the introduction, we aim at solving constructively the inhomogeneous ultrahyperbolic equation

$$(\Delta_x - \Delta_y)\mu = \delta\big(t - |x|\big) \otimes \delta(y) - \delta(x) \otimes \delta\big(t - |y|\big) \quad \text{for } \mu \in \mathcal{S}'(\mathbf{R}^{1+2\nu}). \tag{3}$$

The case of $\nu = 1$ is immediate:

$$\begin{aligned}\big(\partial_x^2 - \partial_y^2\big)\frac{1}{2}Y\big(t - |x| - |y|\big) &= \big[\big(\partial_t^2 - \partial_y^2\big) - \big(\partial_t^2 - \partial_x^2\big)\big]\frac{1}{2}Y\big(t - |x| - |y|\big)\\ &= \delta\big(t - |x|\big) \otimes \delta(y) - \delta(x) \otimes \delta\big(t - |y|\big).\end{aligned}$$

Hence we can assume $\nu > 1$ in the sequel.

Let us define, as usually, the hyperbolic Marcel Riesz kernels $Z_\lambda \in \mathcal{S}'(\mathbf{R}^{\nu+1})$ first, for $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \nu - 1$, by the locally integrable function

$$\begin{aligned}Z_\lambda(t, x) &= \frac{Y(t - |x|)(t^2 - |x|^2)^{(\lambda-\nu-1)/2}}{\pi^{(\nu-1)/2} 2^{\lambda-1} \Gamma(\frac{\lambda}{2}) \Gamma(\frac{\lambda-\nu+1}{2})}\\ &= \frac{Y(t)\chi_+^{\frac{\lambda-\nu-1}{2}}(t^2 - |x|^2)}{\pi^{(\nu-1)/2} 2^{\lambda-1} \Gamma(\frac{\lambda}{2})}\end{aligned}$$

and then, for all $\lambda \in \mathbf{C}$, by analytic continuation with respect to λ, e.g., by using the recurrence relation

$$\Box Z_\lambda = Z_{\lambda-2}$$

(cf. [12, (16), p. 31; (18), p. 32]; [13, (II,3;31), (II,3;33), p. 50]; [9, p. 54]; [4, p. 96, (24)]; [3, (17.9.4.5), (17.9.4.8), p. 267]; [14, ex. 8.4, p. 67]).

For $\lambda = \nu - 1$, we then obtain

$$Z_{\nu-1} = \frac{\delta(t - |x|)}{\pi^{(\nu-1)/2} 2^{\nu-1} \Gamma(\frac{\nu-1}{2}) t} \tag{4}$$

since $\delta(t-|x|)=2tY(t)\delta(t^2-|x|^2)=2tY(t)\chi_+^{-1}(t^2-|x|^2)$. Note that $\chi_+^{-1}=\delta$ can be composed with $t^2-|x|^2$ outside the origin (due to $\mathrm{d}(t^2-|x|^2)\neq 0$ for $t^2=|x|^2,\ (t,x)\neq 0$) and hence (4) holds for $(t,x)\neq 0$. Furthermore, both sides of (4) are homogeneous distributions of degree $-2>-\nu-1$, and hence they coincide in $\mathbf{R}^{\nu+1}$.

Therefore – for space dimensions $\nu>1$ – Eq. (3) can be rewritten as

$$(\Delta_x-\Delta_y)\mu=(4\pi)^{\frac{\nu-1}{2}}\,\Gamma\left(\frac{\nu-1}{2}\right)t\bigl[Z_{\nu-1}(t,x)\otimes\delta(y)-\delta(x)\otimes Z_{\nu-1}(t,y)\bigr].$$

Since, by formula (VII,7;37) in [13, p. 264], $\mathcal{F}(Z_\lambda)=(|x|^2-(t-\mathrm{i}0)^2)^{-\lambda/2}$, Fourier transform yields

$$\begin{aligned}&\bigl(-|x|^2+|y|^2\bigr)\mathcal{F}\mu\\&\quad=(4\pi)^{\frac{\nu-1}{2}}\,\Gamma\left(\frac{\nu-1}{2}\right)\mathrm{i}\partial_t\Bigl[\bigl(|x|^2-(t-\mathrm{i}0)^2\bigr)^{(1-\nu)/2}-\bigl(|y|^2-(t-\mathrm{i}0)^2\bigr)^{(1-\nu)/2}\Bigr]\\&\quad=(4\pi)^{\frac{\nu-1}{2}}\,\Gamma\left(\frac{\nu-1}{2}\right)\frac{\nu-1}{2}(\mathrm{i}\partial_t)\bigl(-|x|^2+|y|^2\bigr)\\&\qquad\times\int_0^1\bigl[s|x|^2+(1-s)|y|^2-(t-\mathrm{i}0)^2\bigr]^{-(1+\nu)/2}\,\mathrm{d}s.\end{aligned}$$

In fact, let us note that

$$F:[0,1]\to\mathcal{S}'\bigl(\mathbf{R}^{1+2\nu}_{t,x,y}\bigr):s\mapsto\bigl[s|x|^2+(1-s)|y|^2-(t-\mathrm{i}0)^2\bigr]^{(1-\nu)/2},$$

is a differentiable distribution-valued function and hence $F(1)-F(0)=\int_0^1 F'(s)\,\mathrm{d}s$. Thus we obtain a solution of (3) by taking

$$\begin{aligned}\mu&=(4\pi)^{\frac{\nu-1}{2}}\,\Gamma\left(\frac{\nu+1}{2}\right)t\int_0^1\mathcal{F}^{-1}\Bigl(\bigl(s|x|^2+(1-s)|y|^2-(t-\mathrm{i}0)^2\bigr)^{-(1+\nu)/2}\Bigr)\,\mathrm{d}s\\&=(4\pi)^{\frac{\nu-1}{2}}\,\Gamma\left(\frac{\nu+1}{2}\right)t\int_0^1 Z_{\nu+1}\left(t,\frac{x}{\sqrt{s}},\frac{y}{\sqrt{1-s}}\right)\bigl[s(1-s)\bigr]^{-\nu/2}\,\mathrm{d}s,\end{aligned}$$

where now $Z_{\nu+1}\in\mathcal{S}'(\mathbf{R}^{2\nu+1})$ is the M. Riesz kernel in the 2ν space variables x,y.

Let us observe that μ is the only solution of (3) such that the intersection of $\operatorname{supp}\mu$ with the strip $[-T,T]\times\mathbf{R}^{2\nu}$ is compact for all $T>0$, for this support property excludes all non-trivial solutions u of the homogeneous equation $(\Delta_x-\Delta_y)u=0$. In fact, if $\psi\in\mathcal{D}((-T,T))$ and $u_\psi\in\mathcal{D}'(\mathbf{R}^{2\nu})$ is defined by $\langle\phi,u_\psi\rangle=\langle\psi(t)\phi(x,y),u\rangle,\ \phi\in\mathcal{D}(\mathbf{R}^{2\nu})$, then the above mentioned support property yields $u_\psi\in\mathcal{E}'(\mathbf{R}^{2\nu}),\ \mathcal{F}(u_\psi)\in\mathcal{C}^\infty(\mathbf{R}^{2\nu})$ and hence $(\Delta_x-\Delta_y)u=0$ implies $(|x|^2-|y|^2)\mathcal{F}u_\psi=0$ and $\mathcal{F}u_\psi=0$ for all ψ, i.e., $u=0$.

In the next section, we shall consider – instead of μ – the distribution-valued function

$$T_\lambda:=\int_0^1\mathcal{F}^{-1}_{t,x,y}\Bigl(\bigl(s|x|^2+(1-s)|y|^2-(t-\mathrm{i}0)^2\bigr)^{-\lambda-\nu-\frac12}\Bigr)\bigl[s(1-s)\bigr]^{\lambda+\frac{\nu}{2}}\,\mathrm{d}s.$$

Evaluating the inverse Fourier transform we have

$$T_\lambda = \int_0^1 Z_{2\lambda+2\nu+1}\left(t, \frac{x}{\sqrt{s}}, \frac{y}{\sqrt{1-s}}\right)\left[s(1-s)\right]^\lambda \,\mathrm{d}s.$$

The above representation of T_λ as an inverse Fourier transform shows its holomorphic dependence on λ for $\operatorname{Re}\lambda > -1-\frac{\nu}{2}$. Furthermore, for $\lambda = -\frac{\nu}{2}$, we obtain $(4\pi)^{\frac{\nu-1}{2}}\,\Gamma(\frac{\nu+1}{2})tT_{-\nu/2} = \mu$.

3. Evaluation of the parameter integral by analytic continuation

For $\operatorname{Re}\lambda > 0$, $T_\lambda \in \mathcal{C}(\mathbf{R}^{1+2\nu})$ can be calculated classically:

$$\begin{aligned} T_\lambda = {} & \frac{1}{\pi^{\nu-\frac12}2^{2\lambda+2\nu}\Gamma(\lambda+\nu+\frac12)\Gamma(\lambda+1)} \\ & \times \int_0^1 \left[s(1-s)t^2-(1-s)|x|^2-s|y|^2\right]^\lambda Y\left(t-\sqrt{\frac{|x|^2}{s}+\frac{|y|^2}{1-s}}\right)\mathrm{d}s. \end{aligned}$$

Let us abbreviate the constant factor in front of the integral by C and consider – for $x \neq 0$, $y \neq 0$ – the function $f(s) = t-\sqrt{\frac{|x|^2}{s}+\frac{|y|^2}{1-s}}$, $0<s<1$. Evidently, f converges to $-\infty$ for $s \searrow 0$ and $s \nearrow 1$. Therefore $T_\lambda(t,x,y)$ vanishes if and only if $f(s_0) = t-|x|-|y|$ is non-positive at the point $s_0 = \frac{|x|}{|x|+|y|}$ where f attains its maximum. In the other case, i.e., if $t > |x|+|y|$, f has the two zeros $s_{1,2} = \frac{t^2+|x|^2-|y|^2\pm\sqrt{A}}{2t^2} \in (0,1)$, wherein

$$\begin{aligned} A(t,x,y) :&= t^4-2t^2\left(|x|^2+|y|^2\right)+\left(|x|^2-|y|^2\right)^2 \\ &= \left(t-|x|-|y|\right)\left(t-|x|+|y|\right)\left(t+|x|-|y|\right)\left(t+|x|+|y|\right), \end{aligned}$$

and thus

$$T_\lambda = CY\left(t-|x|-|y|\right)\int_{s_1}^{s_2}\left[s(1-s)t^2-(1-s)|x|^2-s|y|^2\right]^\lambda \mathrm{d}s.$$

We obtain for the integral

$$\begin{aligned} \int_{s_1}^{s_2} t^{2\lambda}\left[(s_2-s)(s-s_1)\right]^\lambda \mathrm{d}s &= t^{2\lambda}(s_2-s_1)^{2\lambda+1}\frac{\Gamma(\lambda+1)^2}{\Gamma(2\lambda+2)} \\ &= \frac{\Gamma(\lambda+1)^2}{\Gamma(2\lambda+2)}\,t^{-2\lambda-2}A^{\lambda+\frac12} \\ &= \frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{2^{2\lambda+1}\Gamma\left(\lambda+\frac32\right)}\,t^{-2\lambda-2}A^{\lambda+\frac12}. \end{aligned}$$

Thus

$$T_\lambda = \frac{Y(t-|x|-|y|)t^{-2\lambda-2}A^{\lambda+\frac{1}{2}}}{\pi^{\nu-1}2^{4\lambda+2\nu+1}\Gamma(\lambda+\nu+\frac{1}{2})\Gamma(\lambda+\frac{3}{2})}.$$

This computation leads to the following

Proposition 1. *Let* $A = t^4 - 2t^2(|x|^2+|y|^2) + (|x|^2-|y|^2)^2$ *for* $(t,x,y)\in \mathbf{R}^{1+2\nu}$. *Then the unique solution* $\mu \in \mathcal{D}'(\mathbf{R}^{1+2\nu})$ *of*

$$(\Delta_x - \Delta_y)\mu = \delta\bigl(t-|x|\bigr)\otimes\delta(y) - \delta(x)\otimes\delta\bigl(t-|y|\bigr),$$

that has bounded support in x, y *for bounded* t *is given by*

$$\mu = \frac{1}{2}\left(\frac{4}{\pi}\right)^{(\nu-1)/2} Y\bigl(t-|x|-|y|\bigr)\chi_+^{(1-\nu)/2}\left(\frac{A}{t^2}\right). \tag{5}$$

For $\nu = 1, 2$, *the distribution* μ *is a locally integrable function; for* $\nu = 3, 4, \ldots$ *the right-hand side in* (5) *is defined as the value at* $z=\nu$ *of* μ^z *which in turn is defined by*

$$\mu^z = \frac{1}{2}\left(\frac{4}{\pi}\right)^{(z-1)/2} Y\bigl(t-|x|-|y|\bigr)\chi_+^{(1-z)/2}\left(\frac{A}{t^2}\right) \in L^1_{\mathrm{loc}}\bigl(\mathbf{R}^{1+2\nu}\bigr), \quad \operatorname{Re} z < 3,$$

and can analytically be continued into the half-plane $\operatorname{Re} z < 2+\nu$.

Proof. By the definition in Section 2,

$$T_\lambda = \int_0^1 \mathcal{F}^{-1}\Bigl(\bigl(s|x|^2 + (1-s)|y|^2 - (t-\mathrm{i}0)^2\bigr)^{-\lambda-\nu-\frac{1}{2}}\Bigr)\bigl(s(1-s)\bigr)^{\lambda+\frac{\nu}{2}}\,\mathrm{d}s$$

and depends holomorphically on λ for $\operatorname{Re}\lambda > -1-\frac{\nu}{2}$. The calculation above yields

$$T_\lambda = \frac{Y(t-|x|-|y|)}{\pi^{\nu-1}2^{4\lambda+2\nu+1}\Gamma(\lambda+\nu+\frac{1}{2})t}\chi_+^{\lambda+\frac{1}{2}}\left(\frac{A}{t^2}\right) \in \mathcal{C}\bigl(\mathbf{R}^{1+2\nu}\bigr) \quad \text{for } \operatorname{Re}\lambda > 0.$$

Defining μ^z as in the proposition we obtain $\mu^z = (4\pi)^{\nu-\frac{z+1}{2}}\Gamma(\nu-\frac{z-1}{2})tT_{-z/2}$ for $\operatorname{Re} z < 0$. Thus μ^z can be continued holomorphically for $\operatorname{Re} z < 2+\nu$. Due to $\mu = (4\pi)^{\frac{\nu-1}{2}}\Gamma(\frac{\nu+1}{2})tT_{-\nu/2}$ we have $\mu = \mu^\nu$. $\square$

Remark 1. We point out that formula (5) was proven by verification in [8] (cf. Theorem 2.1, p. 380) using a different regularization of μ: L. Hörmander regularizes μ by approximation at the singular values of A, i.e. $|x||y| = 0$, $|x|+|y| = t$, where the pullback of χ_+^λ by A is not well defined, whereas our regularization consists in analytic continuation with respect to z of μ^z into $z=\nu$.

Remark 2. For three space dimensions, i.e. $\nu = 3$, we obtain the Radon-measure $\mu = \frac{t}{4\pi|x||y|}\delta(t-|x|-|y|)$, which – applied to test-functions – appears for the first time in [10].

Remark 3. Since μ is homogeneous (of the degree $1-\nu$), we can restrict it to the hyperplanes $t = R > 0$, and we obtain then solutions $\mu_R := \mu(R, ., .) \in \mathcal{E}'(\mathbf{R}^{2\nu})$ of the equations $(\Delta_x - \Delta_y)\mu_R = \delta(R - |x|) \otimes \delta(y) - \delta(x) \otimes \delta(R - |y|)$. This is the original form of Hörmander's extension of Ásgeirsson's Theorem (cf. [8, Theorem 2.1, p. 380]).

4. Definition of μ by a recurrence relation

From Proposition 1 we know that the function $z \mapsto \mu^z \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+2\nu})$, $\operatorname{Re} z < 3$, can analytically be continued to the half-plane where $\operatorname{Re} z < 2 + \nu$. That was a consequence of the explicit representation of $\mathcal{F}\mu^z$ by a parameter integral. In order to give a more direct definition of $\mu_t = \mu^\nu(t, ., .) \in \mathcal{E}'(\mathbf{R}^{2\nu})$ (cf. Remark 3), let us, for fixed $t > 0$, extend analytically the function $z \mapsto \mu_t^z := \mu^z(t, ., .)$ by a recurrence relation.

Using the formula

$$(\Delta_x - \Delta_y) f(A) = 8\bigl(|x|^2 - |y|^2\bigr)\bigl[(\nu + 1) f'(A) + 2A f''(A)\bigr]$$

for arbitrary $f \in \mathcal{C}^2(\mathbf{R})$ and $A = t^4 - 2t^2(|x|^2 + |y|^2) + (|x|^2 - |y|^2)^2$ as above (cf. [8, p. 379]), we obtain

$$\begin{aligned}
&(\Delta_x - \Delta_y) Y\bigl(t - |x| - |y|\bigr) A^{(1-z)/2} \\
&\quad = 4(1 - z)(\nu - z)\bigl(|x|^2 - |y|^2\bigr) Y\bigl(t - |x| - |y|\bigr) A^{-(1+z)/2}
\end{aligned}$$

if $\operatorname{Re} z < -3$. Since

$$\mu_t^z = \frac{1}{2}\left(\frac{4}{\pi}\right)^{(z-1)/2} \frac{t^{z-1}}{\Gamma(\frac{3-z}{2})} Y\bigl(t - |x| - |y|\bigr) A^{(1-z)/2} \in L^1\bigl(\mathbf{R}^{2\nu}\bigr) \quad \text{for } \operatorname{Re} z < 3,$$

we conclude that

$$(\Delta_x - \Delta_y)\mu_t^z = \frac{2\pi}{t^2}(\nu - z)\bigl(|x|^2 - |y|^2\bigr)\mu_t^{z+2} \tag{6}$$

first for $\operatorname{Re} z < -3$ and then, by analytic continuation, for $\operatorname{Re} z < \nu$, where both sides of (6) are holomorphic according to Proposition 1.

In $U := \{(x, y) \in \mathbf{R}^{2\nu};\ |x| \neq |y|\}$, the distribution-valued function

$$\{z \in \mathbf{C};\ \operatorname{Re} z < \nu\} \to \mathcal{E}'(U) : z \mapsto \mu_t^z\big|_U$$

can be extended meromorphically to the whole z-plane by (6). Due to the factor $\nu - z$ on the right-hand side of (6), $\mu_t^z|_U$ has simple poles in $z = \nu + 2, \nu + 4, \ldots$.

In a neighborhood of the conic $|x|^2 - |y|^2 = 0$, the pullback $\chi_+^{(1-z)/2}(A)$ is well-defined for all $z \in \mathbf{C}$, since there either $A \neq 0$ or $\mathrm{d}A \neq 0$, cf. [5, p. 70]; [7, Theorem 6.1.2, p. 134]. Note also that $Y(t - |x| - |y|)\chi_+^{(1-z)/2}(A)$ coincides with $\chi_+^{(1-z)/2}(A)$ in such a neighborhood. Thus we obtain:

Proposition 2. *For fixed $t > 0$ and $z \in \mathbf{C}$ with $\operatorname{Re} z < 3$, define*

$$\mu_t^z := \frac{1}{2}\left(\frac{4}{\pi}\right)^{(z-1)/2} Y\bigl(t - |x| - |y|\bigr)\chi_+^{(1-z)/2}\left(\frac{A}{t^2}\right) \in L^1\bigl(\mathbf{R}^{2\nu}\bigr)$$

with $A = t^4 - 2t^2(|x|^2 + |y|^2) + (|x|^2 - |y|^2)^2$. *Then the distribution-valued function*

$$\{z \in \mathbf{C};\ \operatorname{Re} z < 3\} \to \mathcal{E}'\big(\mathbf{R}^{2\nu}\big) : z \mapsto \mu_t^z$$

extends holomorphically to $\mathbf{C} \setminus \{\nu + 2k;\ k \in \mathbf{N}\}$. *This extended function* μ_t^z *has simple poles in* $z = \nu + 2k,\ k \in \mathbf{N}$, *and, in particular,*

$$\operatorname*{Res}_{z=\nu+2} \mu_t^z = -\frac{1}{2\pi}\big[\delta\big(t - |x|\big) \otimes \delta(y) + \delta(x) \otimes \delta\big(t - |y|\big)\big].$$

Proof. The validity of the analytic extension has been shown in the discussion leading to Proposition 2. In order to check the statement on the poles of μ_t^z, we can restrict the calculation to $\mathcal{E}'(U)$. By means of (6) and Proposition 1 we obtain

$$\begin{aligned}
\operatorname*{Res}_{z=\nu+2} \mu_t^z\big|_U &= \lim_{z\to\nu} (z-\nu)\mu_t^{z+2}\big|_U \\
&= -\frac{t^2}{2\pi(|x|^2 - |y|^2)}\,(\Delta_x - \Delta_y)\mu_t^\nu\big|_U \\
&= -\frac{t^2}{2\pi(|x|^2 - |y|^2)}\big[\delta\big(t - |x|\big) \otimes \delta(y) - \delta(x) \otimes \delta\big(t - |y|\big)\big]\Big|_U \\
&= -\frac{1}{2\pi}\big[\delta\big(t - |x|\big) \otimes \delta(y) + \delta(x) \otimes \delta\big(t - |y|\big)\big]\Big|_U .
\end{aligned}$$

Similarly, for $k = 2, 3, \ldots,$

$$\begin{aligned}
\operatorname*{Res}_{z=\nu+2k} \mu_t^z\big|_U &= \lim_{z\to\nu+2k-2} (z - \nu - 2k + 2)\mu_t^{z+2}\big|_U \\
&= \frac{t^2}{2\pi(2-2k)(|x|^2 - |y|^2)} \lim_{z\to\nu+2k-2} (z - \nu - 2k + 2)(\Delta_x - \Delta_y)\mu_t^z\big|_U
\end{aligned}$$

and hence

$$\operatorname*{Res}_{z=\nu+2k} \mu_t^z = -\frac{t^2}{4\pi(k-1)(|x|^2 - |y|^2)}(\Delta_x - \Delta_y) \operatorname*{Res}_{z=\nu+2k-2} \mu_t^z, \quad k = 2, 3, \ldots$$

is a non-vanishing distribution with support in the set $\{(x, y) \in \mathbf{R}^{2\nu};\ |x||y| = 0, |x| + |y| = t\}$. $\square$

Remark. The evaluation of $\operatorname{Res}_{z=\nu+2}\mu_t^z$ in the above proof is based on the fact that $\mu_t = \mu_t^z$ solves (3) for fixed t. Conversely, the solution of (3) could be deduced from a direct calculation of this residue, e.g., by introducing coordinates in a similar way as in [8, p. 379].

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